FREEDOM OF $h(2)$-VARIATIONALITY AND METRIZABILITY OF SPRAYS

S.G. ELGENDI AND Z. MUZSNAY

Abstract. In this paper we are investigating variational homogeneous second order differential equations by considering the questions of how many different variational principles exist for a given spray. We focus our attention on $h(2)$-variationality; that is, the regular Lagrange function is homogeneous of degree two in the directional argument. Searching for geometric objects characterizing the degree of freedom of $h(2)$-variationality of a spray, we show that the holonomy distribution generated by the tangent direction to the parallel translations can be used to calculate it. As a working example, the class of isotropic sprays is considered.

1. Introduction

In 1960, W. Ambrose et al. [1] introduced the notion of sprays to give an intrinsic presentation of second order ordinary differential equations. All sprays are associated with a second order system of ordinary differential equations and conversely, a spray can be associated with a second order system of ordinary differential equations (SODE). In the most interesting cases the spray can be derived from a variational principle. A particular class of variational sprays is composed by the metrizable sprays. These sprays or SODEs can be viewed as the geodesic equations of a metric. Several papers are devoted to the inverse problem of the calculus of variations and in particular, to the metrizability problem (see for example [2, 6, 8, 12] and [3, 5, 9, 10, 11, 16] respectively). In this paper we are considering a different aspect of the problem which is motivated by the fact that there are sprays for which

(a) there is no regular Lagrange function, that is, the spray is not variational,
(b) there is essentially a unique Lagrange function,
(c) there are several different regular Lagrange functions.

The questions of *how many different Lagrange functions* can be associated with a spray and *how to determine* this number in terms of geometric objects are very interesting because the answers can lead to a better understanding of the structure. In this paper we propose to investigate the above questions by considering the Euler-Lagrange partial differential system associated to sprays. In Definition 2.6 we introduce the notion of *variational freedom*, denoted by $\mathcal{V}_S$, which shows how many different variational principle can be associated to the spray or in other words, how many essentially different regular Lagrange functions exist for a given spray. In general a spray S is non-variational and therefore $\mathcal{V}_S = 0$. For most of the variational cases there is a unique variational principle admitting S as a solution, that is $\mathcal{V}_S = 1$. It may also happen that $\mathcal{V}_S > 1$, that is, there exist $\mathcal{V}_S$ essentially different Lagrange functions and $\mathcal{V}_S$ essentially different variational principle associated to S.

It is particularly interesting when the Lagrange functions are homogeneous. This is the case for many examples: in general reativity, Riemannian geometry, Finslerian geometry, etc. That motivates the problem to investigate the freedom of $h(k)$-variationality, when the Lagrange function must be k-homogeneous. In this paper we show that in the regular case the *holonomy distribution* can be used to determine $\mathcal{V}_{S,2}$. In Section 4 we give an explicit formula how $\mathcal{V}_{S,2}$ can be calculated. As a working example we consider the class of isotropic sprays in Section 5.

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2. Preliminaries

In this section, we give a brief introduction that will be needed throughout. For more details, we refer to [7, 8]. Let M be an n-dimensional manifold, (TM, π_M, M) be its tangent bundle and $(\mathcal{T}M, \pi, M)$ the subbundle of nonzero tangent vectors. We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM. The vector 1-form J on TM locally defined by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called the natural almost-tangent structure of TM. The vertical vector field $\mathcal{C} = y^i \frac{\partial}{\partial y^i}$ on TM is called the canonical or Liouville vector field. Recall that $\mathcal{C}$ is the infinitesimal generator of the one-parameter group of (positive) homoteties. A vector ℓ-form L on TM is homogeneous of degree r if $d_{\mathcal{C}} L = (r-1)L$. A scalar p-form ω on TM is homogeneous of degree r if $\mathcal{L}_{\mathcal{C}} \omega = r\omega$. In particular a function $E \in C^\infty(\mathcal{T}M)$ is k-homogeneous if

$$\mathcal{L}_{\mathcal{C}} E = kE. \tag{2.1}$$

Semi-sprays and sprays. A vector field $S \in \mathfrak{X}(\mathcal{T}M)$ is called a semi-spray if $JS = \mathcal{C}$. If in addition $[\mathcal{C}, S] = S$ then S is called homogeneous semi-spray, or simply a spray. Locally, a semi-spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}. \tag{2.2}$$

where the functions $G^i = G^i(x, y)$ are the *coefficients* of the semi-spray. If S is a spray, then the *coefficients* are homogeneous functions of degree 2 in the $y = (y^1, \dots, y^n)$ variable. A curve $c : I \to M$ is called is called *geodesic* of a semi-spray S if $S \circ c' = c''$. Locally, $c(t) = (x^i(t))$ is a geodesic of (2.2) if and only if it satisfies the equation

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0, \tag{2.3}$$

therefore semi-sprays can be seen as the coordinate-free version of system of second order differential equations.

Definition 2.1. A *regular Lagrange function* $E \colon TM \to \mathbb{R}$ is continuous, smooth on $\mathcal{T}M$, and the matrix field

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} \tag{2.4}$$

is regular on $\mathcal{T}M$. The Lagrange function E is called k-homogeneous if it is homogeneous of degree k in the directional argument $y = (y^i)$. A regular $2-$homogeneous Lagrange function E is called *Finsler energy function* if (2.4) is positive definite.

If E is a regular Lagrange function, then the 2-form $\Omega_E := dd_J E$ is non-degenerate and the Euler-Lagrange equation

$$i_S \Omega_E + d(\mathcal{L}_{\mathcal{C}} E - E) = 0 \tag{2.5}$$

uniquely determines a semi-spray S. This semi-spray is called the *geodesic semi-spray* of E.

Let us consider the *inverse problem*: A semi-spray S on a manifold M is called *variational* if there exists a variational principle, that is a regular Lagrange function $E \colon TM \to \mathbb{R}$, such that the stationary curves of the functional

$$I(\gamma) = \int E(\gamma(t), \dot\gamma(t)) dt$$

are the geodesic curves of the semi-spray S. It is particularly interesting, when both the semi-spray and the regular Lagrange function are homogeneous. This is the case for many important examples: in general relativity, Riemannian geometry, Finslerian geometry, etc. That motivates the following

Definition 2.2. A given spray S on a manifold M is called $h(k)-$*variational* ($k \in \mathbb{N}$) if there exists a k-homogeneous regular Lagrange function $E \colon TM \to \mathbb{R}$ whose geodesic spray coincide with S. In particular S is *metrizable*, if there exists a Finsler energy function such that the associated geodesic spray is S.

J. Szenthe proved that in the analytical case if there exists a regular associated Lagrangian for a spray S, then there exists a $2-$homogeneous regular associated Lagrangian too. This result can be reformulated as

Proposition 2.3 (Szenthe [14])**.** *An analytical spray on an analytical manifold is variational if and only if it is $h(2)-$variational.*

To decide whether or not a spray S can be derived from a variational principle one has to consider the Euler-Lagrange partial differential equation which is formally the same as (2.6) but considered this time as a differential system on the unknown Lagrange function E.

Definition 2.4. The solutions of (2.6) are called *Euler-Lagrange functions* of the spray S. The set of Euler-Lagrange functions of S will be denoted by $\mathcal{E}_S$. The subset of k-homogeneous Euler-Lagrange functions will be denoted by $\mathcal{E}_{S,k}$.

Let S be a spray on a manifold M. The *Euler-Lagrange form* associated with E is $\omega_E := i_S\Omega_E + d\mathcal{L}_C E - dE$. Then the Euler-Lagrange PDE equation (2.5) can be written as

$$\omega_E = 0. \tag{2.6}$$

Taking the homogeneity condition (2.1) into account we get that

$$\mathcal{E}_{S,k} = \{ E \in C^\infty(\mathcal{T}M) \mid \omega_E{=}0,\ \mathcal{L}_{\mathcal{C}}E{=}kE \} . \tag{2.7}$$

To summarize the formalism we get the

Property 2.5. A semi-spray S is variational if there exists a regular function $E \in \mathcal{E}_S$. A spray S is $h(k)$-variational if there exists a regular function $E \in \mathcal{E}_{S,k}$.

Several works are devoted to the inverse problem of the calculus of variations (see for example [2, 6, 8, 12] and the references therein). In this paper we are considering a different aspect of this problem: *How many variational principles* exist for a given spray and how this number can be determined in terms of *geometric objects* associated to the spray? To formulate the problem in a precise way we introduce the following

Definition 2.6. Let S be a semi-spray. If S is variational, then its *variational freedom* is $\mathcal{V}_S (\in \mathbb{N})$ where $\mathcal{V}_S = rank\,(\mathcal{E}_S)$. If S is non-variational, then we set $\mathcal{V}_S = 0$.

We precise here that the notation $\mathcal{V}_S = rank\,(\mathcal{E}_S)$ means that $\mathcal{E}_S$ can be locally generated by its $\mathcal{V}_S$ functionally independent of its elements. In other words, if the variational freedom of S is $\mathcal{V}_S \geq 1$ then for every $v_0 \in \mathcal{T}M$ there exists a neighbourhood $U \subset \mathcal{T}M$ and $E_1, ..., E_{\mathcal{V}_S} \in \mathcal{E}_S$ functionally independent on U such that any $E \in \mathcal{E}_S$ can be expressed as

$$E(v) = \varphi\big(E_1(v), \dots, E_{\mathcal{V}_S}(v)\big), \qquad \forall\ v \in U,$$

with some function $\varphi \colon \mathbb{R}^{\mathcal{V}_S} \to \mathbb{R}$.

For (homogeneous) spray, we can consider the following analogous

Definition 2.7. For a variational spray S the $h(k)$-variational freedom is $\mathcal{V}_{S,k}$ if $\mathcal{V}_{S,k} = rank\,(\mathcal{E}_{S,k})$. We set $\mathcal{V}_{S,k} = 0$ if there is no regular element in $\mathcal{E}_{S,k}$.

We remark, that a 1-homogeneous Lagrange function cannot be regular, therefore for any sprays S we have $\mathcal{V}_{S,1} = 0$.

It is particularly interesting the $\mathcal{V}_{S,2} = rank\,(\mathcal{E}_{S,2})$ which is showing how many different $2-$homogeneous Lagrange functions or variational principles exist for the given spray. As it was already mentioned, in many applications (general relativity, in Riemannian and Finslerian geometry, etc) the energy function must be 2-homogeneous. This is why in this paper we are focusing our attention on this special case.

In Section 3 we introduce the most important geometric objects (parallel translation, holonomy distribution, holonomy invariant functions) needed to compute $\mathcal{V}_{S,2}$ and in Section 4 we determine $\mathcal{V}_{S,2}$ in the case when the associated parallel translation is regular.

3. Holonomy invariant functions

Connection, parallel translation.
Every semi-spray S induces an Ehresmann connection (see [15]). The corresponding decomposition is $T\mathcal{T}M = H\mathcal{T}M \oplus V\mathcal{T}M$, where $V\mathcal{T}M = \operatorname{Ker}\pi_*$ is the vertical and $H\mathcal{T}M$ is the horizontal subbundle of $T\mathcal{T}M$ defined through the corresponding *horizontal and vertical projectors.* Locally, the projectors associated to (2.2) can be expressed as $h = \frac{\delta}{\delta x^i} \otimes dx^i$ and $v = \frac{\partial}{\partial y^i} \otimes \delta y^i$ where $\delta y^i = dy^i + N^i_j dx^j$,

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \tag{3.1}$$

and $N^j_i = \frac{\partial G^j}{\partial y^i}$. The modules of horizontal and vertical vector fields will be denoted by $\mathfrak{X}^h(TM)$ and $\mathfrak{X}^v(TM)$ respectively.

The *parallel translation* of a vector along curves is defined through horizontal lifts: Let $\gamma\colon [0,1] \to M$ be a curve such that $\gamma(0) = p$ and $\gamma(1) = q$. Let $\gamma^h(0) = v$, $\gamma^h(1) = w$ and γ^h be a horizontal lift of the curve γ, that is $\pi \circ \gamma^h = \gamma$ and $\dot{\gamma}^h(t) \in H_{\gamma^h(t)}$. The parallel translation $\tau\colon T_pM \to T_qM$ along γ is defined as follows: $\tau(v) = w$.

The curvature tensor $R = -\frac{1}{2}[h,h]$ of the nonlinear connection satisfies

$$R(X,Y) = -v[hX, hY], \tag{3.2}$$

and characterizes the integrability of the horizontal distribution: $H\mathcal{T}M$ is integrable if and only if the curvature is identically zero. In a local coordinate system, the curvature is given by $R = R^i_{jk} dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i}$ where $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}$.

Holonomy distribution.

Definition 3.1 ([11])**.** The *holonomy distribution* $\mathcal{D}_{\mathcal{H}}$ of a spray S is the distribution on TM generated by the horizontal vector fields and their successive Lie-brackets, that is

$$\mathcal{D}_{\mathcal{H}} := \Big\langle \mathfrak{X}^h(TM) \Big\rangle_{Lie} = \Big\{ [X_1, [\ldots[X_{m-1}, X_m]\ldots]] \;\Big|\; X_i \in \mathfrak{X}^h(TM) \Big\} \tag{3.3}$$

Remark 3.2. The holonomy distribution $\mathcal{D}_{\mathcal{H}}$ is the smallest involutive distribution containing the horizontal distribution. Using the horizontal and vertical projectors we have

$$\mathcal{D}_{\mathcal{H}} = h(\mathcal{D}_{\mathcal{H}}) \oplus v(\mathcal{D}_{\mathcal{H}}) = H\mathcal{T}M \oplus v(\mathcal{D}_{\mathcal{H}}).$$

From (3.2) we can get that the image of the curvature tensor is a subset of the vertical part of the holonomy distribution, that is $\operatorname{Im} R \subset v(\mathcal{D}_{\mathcal{H}})$, and we have $\mathcal{D}_{\mathcal{H}} = H\mathcal{T}M$ if and only if $R \equiv 0$.

Remark 3.3. When $\mathcal{D}_{\mathcal{H}}$ is a regular distribution, then it is integrable. Using the definition of parallel translation via horizontal lifts it is easy to see that the integral manifold through $v \in TM$, $\mathcal{O}_\tau(v)$, is the orbit of v with respect to all possible parallel translations. By Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v \in \mathcal{T}M$ such that the components of $\mathcal{O}_\tau \cap U$ are the sets

$$\{w \in U \mid z^i(w) = z^i_0, \ \dim \mathcal{O}_\tau + 1 \le i \le 2n\}, \qquad |z^i_0| < \epsilon. \tag{3.4}$$

Definition 3.4. We say that the parallel translation is *regular* if the distribution $\mathcal{D}_{\mathcal{H}}$ is regular and the orbits of the parallel translation are regular in the sense that and for any $v \in \mathcal{T}M$ there is a neighbourhood $U \subset \mathcal{T}M$ such that any orbits $\mathcal{O}_\tau$ have at most one connected component in U.

If the parallel translation is regular, then there exists a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of any $v \in \mathcal{T}M$ such that in (3.4) different z^i coordinates ($\dim \mathcal{O}_\tau + 1 \le i \le 2n$) correspond to different orbits of the parallel translation.

Holonomy invariant functions.

Definition 3.5. Let S be a spray. A function $E \in C^\infty(TM)$ is called *holonomy invariant*, if it is invariant with respect to parallel translation, that is, for any $v \in TM$ and for any parallel translation τ we have $E(\tau(v)) = E(v)$. The set of holonomy invariant functions will be denoted by $\mathcal{H}_S$.

In the case when the parallel translation is regular, the tangent spaces of its orbits are given by the holonomy distribution $\mathcal{D}_\mathcal{H}$, that is $T_v\big(O_\tau(v)\big) = \mathcal{D}_\mathcal{H}(v)$. Consequently, $E \in C^\infty(TM)$ is a holonomy invariant function if and only if we have $\mathcal{L}_X E = 0$, $X \in \mathcal{D}_\mathcal{H}$ that is

$$\mathcal{H}_S = \{E \in C^\infty(\mathcal{T}M) \mid \mathcal{L}_X E = 0,\ X \in \mathcal{D}_\mathcal{H}\}\,. \tag{3.5}$$

The subset of k-homogeneous holonomy invariant functions will be denoted by $\mathcal{H}_{S,k}$. Using (2.1) we get

$$\mathcal{H}_{S,k} = \{E \in \mathcal{H}_S \mid \mathcal{L}_\mathcal{C} E = kE\}\,. \tag{3.6}$$

4. Euler-Lagrange functions and $h(2)$-variational freedom of sprays

We can observe the following

Property 4.1. The $\mathcal{E}_S$ and $\mathcal{E}_{S,k}$ ($k \in \mathbb{N}$) are vector spaces over $\mathbb{R}$.

Proof. Both the Euler-Lagrange equation (2.5) and the homogeneity equation (2.1) linear parial differential equations. Therefore linear combination of their solutions with constant coefficients are also solutions. □

In particular, Property 4.1 states that linear combination of 2-homogeneous Euler-Lagrange functions of S are also 2-homogeneous Euler-Lagrange functions of S. We can consider this combination as a *trivial combination* of Euler-Lagrange functions. As the next proposition shows, a much wider combination of homogeneous Euler-Lagrange functions can produce new homogeneous Euler-Lagrange functions:

Proposition 4.2. *A* 1*-homogeneous functional combination of* 2*-homogeneous Euler-Lagrange functions of a spray S is also a* 2*-homogeneous Euler-Lagrange functions of S.*

To prove the proposition we will use the following

Lemma 4.3. *A 2-homogeneous Lagrangian is an Euler-Lagrange function of a spray S if and only if it is a holonomy invariant function. Using the notation* (2.7) *and* (3.6) *we have*

$$\mathcal{E}_{S,2} = \mathcal{H}_{S,2}. \tag{4.1}$$

Proof. Let $\mathfrak{h}\colon TTM \to \mathcal{D}_\mathcal{H}$ be an arbitrary projection on $\mathcal{D}_\mathcal{H}$. In [11, p. 86, Theorem 1.] it was proven that a 2-homogeneous Lagrange functions $E\colon \mathcal{T}M \to \mathbb{R}$ is a solution of the Euler-Lagrange PDE if and only if it satisfies the equation

$$d_\mathfrak{h} E = 0, \tag{4.2}$$

where the $d_\mathfrak{h}$ operator is defined by the formula $d_\mathfrak{h}E(X) = \mathfrak{h}X(E) = \mathcal{L}_{\mathfrak{h}X}E$. Consequently (4.2) is satisfied if and only if E is a holonomy invariant function. □

Proof of Proposition 4.2. Let $\varphi = \varphi(z_1, \dots, z_r)$ be a smooth 1-homogeneous function and consider the functional combination

$$E := \varphi\big(E_1, \dots, E_r\big). \tag{4.3}$$

of $E_1, \dots, E_r \in \mathcal{E}_{S,2}$, that is, 2-homogeneous Euler-Lagrange functions of a spray S. To prove the theorem we have to show that E is also a $2-$homogeneous Euler-Lagrange function of S. Then, because of the $1-$homogeneity of φ and the $2-$homogeneity of E_i, $i = 1, \dots, r$, we have

$$E(x, \lambda y) = \varphi(E_1(x, \lambda y), ..., E_r(x, \lambda y)) = \varphi(\lambda^2 E_1(x, y), ..., \lambda^2 E_r(x, y)) = \lambda^2 E(x, y),$$

hence E is 2−homogeneous. Moreover, using (4.1) we have $E_i \in \mathcal{H}_{S,2}$ and from (3.5) we get $\mathcal{L}_X E_i = 0$ for any vector field $X \in \mathcal{D}_{\mathcal{H}}$ in the holonomy distribution. Consequently, for $X \in \mathcal{D}_{\mathcal{H}}$ we have

$$\mathcal{L}_X E = \frac{\partial \varphi}{\partial z^1} \cdot \mathcal{L}_X E_1 + \cdots + \frac{\partial \varphi}{\partial z^r} \cdot \mathcal{L}_X E_r = 0,$$

which shows that $E \in \mathcal{H}_{S,2}$ and from (4.1) we get $E \in \mathcal{E}_{S,2}$. □

Proposition 4.2 shows that functional combination of Euler-Lagrange functions for the spray can generate new Euler-Lagrange functions, hence new variational principles for a spray. The variational freedom introduced in Definition 2.6 tells us how many essentially different variational principles exist for a given spray. The following Theorem can be used to determine the $h(2)$-variational freedom in terms of geometric quantities associated to the spray.

Theorem 4.4. *Let S be a metrizable spray such that the parallel translation with respect to the associated connection is regular. Then*

$$\mathcal{V}_{S,2} = \operatorname{codim} \mathcal{D}_{\mathcal{H}}. \tag{4.4}$$

To prove the above theorem we need the following lemmas:

Lemma 4.5. *Let S be a spray and $E_o \in \mathcal{E}_{S,2}$ nonzero on $\mathcal{T}M$. Then E is a 2−homogeneous Euler-Lagrange function of S if and only if $\theta := E/E_o$ is a 0−homogeneous holonomy invariant function:*

$$E \in \mathcal{E}_{S,2} \quad \Longleftrightarrow \quad \theta = E/E_o \in \mathcal{H}_{S,0}$$

Proof. Using Lemma 4.3 we obtain that both E and E_o are 2-homogeneous holonomy invariant functions. Thus, $\theta := E/E_o$ is a 0-homogeneous holonomy invariant function, that is, $\theta \in \mathcal{H}_{S,0}$. Conversely, assume that $\theta = E/E_o \in \mathcal{H}_{S,0}$. Then $E = \theta E_o$ is 2−homogeneous holonomy invariant function. By Lemma 4.3, E is an Euler-Lagrange function of the spray S. □

Let us consider the smallest involutive distribution contained $\mathcal{D}_{\mathcal{H}}$ and the Liouville vector field $\mathcal{C}$:

$$\mathcal{D}_{\mathcal{H},C} := \left\langle \mathcal{D}_{\mathcal{H}}, \mathcal{C} \right\rangle_{Lie}.$$

We have the following

Lemma 4.6. *If S is a spray, then $\mathcal{D}_{\mathcal{H},C}$ is linearly generated by $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{C}$, that is,*

$$\mathcal{D}_{\mathcal{H},C} = Span\{\mathcal{D}_{\mathcal{H}}, \mathcal{C}\}. \tag{4.5}$$

Proof. If $\mathcal{C} \in \mathcal{D}_{\mathcal{H}}$ then $\mathcal{D}_{\mathcal{H},C} = \mathcal{D}_{\mathcal{H}}$ and (4.5) is true. Let us consider the case when $\mathcal{C} \notin \mathcal{D}_{\mathcal{H}}$. Take $X, Y \in \mathcal{D}_{\mathcal{H},C}$. Using the decomposition $X = X_{\mathcal{D}_{\mathcal{H}}} + X_{\mathcal{C}}$ and $Y = Y_{\mathcal{D}_{\mathcal{H}}} + Y_{\mathcal{C}}$ corresponding to the directions $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{C}$ we get

$$[X, Y] = [X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_{\mathcal{C}}, Y_{\mathcal{C}}] + [X_{\mathcal{C}}, Y_{\mathcal{D}_{\mathcal{H}}}] + [X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{C}}]. \tag{4.6}$$

We have $[X_{\mathcal{C}}, Y_{\mathcal{C}}] \in Span\{\mathcal{C}\}$ and because of the involutivity of $\mathcal{D}_{\mathcal{H}}$ we have also $[X_{\mathcal{D}_{\mathcal{H}}}, Y_{\mathcal{D}_{\mathcal{H}}}] \in \mathcal{D}_{\mathcal{H}}$. Let us consider a local basis $\mathcal{B} = \left\{ \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right\}$ of the horizontal space $H\mathcal{T}M$ introduced in (3.1). Then the holonomy distribution $\mathcal{D}_{\mathcal{H}}$ can be generated locally by the elements of $\mathcal{B}$ and by their successive Lie brackets. Since the spray coefficients $G^i(x, y)$ introduced in (2.2) are 2-homogeneous in the y-variables, we have $[\mathcal{C}, \frac{\delta}{\delta x^i}] = 0$. By the Jacobi identity, this is also true for the successive brackets of the $\frac{\delta}{\delta x^i}$'s. Now, $Y_{\mathcal{D}_{\mathcal{H}}} \in \mathcal{D}_{\mathcal{H}}$ can be written as a linear combination of the elements $Y_{\mathcal{D}_{\mathcal{H}}} = g^{\alpha} Y_{\alpha}$, where $Y_{\alpha} \in \mathcal{D}_{\mathcal{H}}$ can be obtained by successive brackets of the $\frac{\delta}{\delta x^i}$'s, and therefore $[\mathcal{C}, Y_{\alpha}] = 0$. Hence, for the $\mathcal{C}$-directional component of X we have $X_{\mathcal{C}} = X^c \mathcal{C}$ with $X^c \in C^{\infty}(\mathcal{T}M)$ and

$$[X_{\mathcal{C}}, Y_{\mathcal{D}_{\mathcal{H}}}] = [X^c \mathcal{C}, g^{\alpha} Y_{\alpha}] = (X_{\mathcal{C}} g^{\alpha}) Y_{\alpha} - (Y_{\mathcal{D}_{\mathcal{H}}} X^c) \mathcal{C} + X^c g^{\alpha} [\mathcal{C}, Y_{\alpha}] = (X_{\mathcal{C}} g^{\alpha}) Y_{\alpha} - (Y_{\mathcal{D}_{\mathcal{H}}} X^c) \mathcal{C}$$

which is an element of $Span\{\mathcal{D}_{\mathcal{H}}, \mathcal{C}\}$. The same argument is valid for the fourth term in (4.6). □

Lemma 4.7. *If the spray S is metrizable then $\mathcal{C}$ is transverse to $\mathcal{D}_{\mathcal{H}}$ on $\mathcal{T}M$, that is*

$$Span\{\mathcal{D}_{\mathcal{H}}, \mathcal{C}\} = \mathcal{D}_{\mathcal{H}} \oplus Span\{\mathcal{C}\}. \tag{4.7}$$

Proof. If S is metrizable, then there exists a Finsler energy function $E_o \in \mathcal{E}_{S,2}$ of S. Because of Proposition 4.3 we have $E_o \in \mathcal{H}_{S,2}$. On the other hand, by using the homogeneity property of E_o we have $\mathcal{L}_{\mathcal{C}_v} E = 2E(v) > 0$ at any point $v \in \mathcal{T}M$. But the derivatives of E_o with respect to the elements of $\mathcal{D}_{\mathcal{H}}$ is zero. Therefore we obtain that $\mathcal{C} \notin \mathcal{D}_{\mathcal{H}}$ at $v \in \mathcal{T}M$. Using Lemma 4.6 we get (4.7). □

Proof of Theorem 4.4. Let us denote by $\kappa(\in \mathbb{N})$ the dimension of $\mathcal{D}_{\mathcal{H}}$. We will show in the proof that in a neighbourhood of any $v \in \mathcal{T}M$ one can find exactly $\operatorname{Codim} \mathcal{D}_{\mathcal{H}} = 2n - \kappa$ locally functionally independent elements in $\mathcal{E}_{S,2}$.

As the spray S is metrizable, therefore there exists a Finsler energy function $E_o \in \mathcal{E}_{S,2}$ associated to S. From (4.5) and (4.7), we have $\mathcal{D}_{\mathcal{H},C} = \mathcal{D}_{\mathcal{H}} \oplus \mathcal{C}$ end therefore $\dim \mathcal{D}_{\mathcal{H},C} = \kappa + 1$. Both $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{H},C}$ are involutive smooth distributions on $\mathcal{T}M$. By Frobenius integrability theorem one can find a coordinate system (U, z) of $\mathcal{T}M$ in a neighborhood of $v_0 \in \mathcal{T}M$, such that $z^i(v) = 1$, $z(U) =]1-\epsilon, 1+\epsilon[^{2n}$ and for all $z_0^{\kappa+1}, ..., z_0^{2n}$ with $|1-z_0^i| < \epsilon$, the sets

$$\mathcal{O}_\tau = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+1 \le i \le 2n\}, \qquad \mathcal{N} = \{w \in U \mid z^i(w) = z_0^i,\ \kappa+2 \le i \le 2n\}$$

are integral manifolds of the distributions $\mathcal{D}_{\mathcal{H}}$ respectively $\mathcal{D}_{\mathcal{H},C}$ over U. Moreover, by the regularity of the parallel translation the coordinate neighbourhood U can be choosen such a way that for any $v \in U$ the orbit $\mathcal{O}_\tau(v)$ of v under the parallel translations has only one component in U. In this case different z^i coordinates $(\kappa + 1 \le i \le 2n)$ correspond to different orbits, hence these coordinates parametrise the orbits of the parallel translations on U. Let

$$\mathcal{D}_{\mathcal{H}} = Span \left\{ \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^\kappa} \right\}, \qquad \mathcal{D}_{\mathcal{H},C} = Span \left\{ \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^\kappa}, \frac{\partial}{\partial z^{\kappa+1}} \right\} \tag{4.8}$$

where $Span \left\{ \frac{\partial}{\partial z^{\kappa+1}} \right\} = Span \{\mathcal{C}\}$, that is, $\frac{\partial}{\partial z^{\kappa+1}} = \lambda \mathcal{C}$, with $\lambda(v_0) \neq 0$. Hence, from (2.1) we get

$$\frac{\partial E_o}{\partial z^{\kappa+1}}(v_0) = \lambda (\mathcal{C} E_o)(v_0) = 2\lambda E_o(v_0) \neq 0. \tag{4.9}$$

Considering the set of $0-$homogeneous holonomy invariant functions, we have

$$\theta \in \mathcal{H}_{S,0} \iff \left\{ \begin{array}{l} \mathcal{L}_X \theta = 0,\ \forall\, X \in \mathcal{D}_{\mathcal{H}} \\ \mathcal{L}_{\mathcal{C}} \theta = 0, \end{array} \right\} \iff \mathcal{L}_X \theta = 0,\ \forall\, X \in \mathcal{D}_{\mathcal{H},C}. \tag{4.10}$$

From (4.8) and from (4.10), it follows that $\theta \in \mathcal{H}_{S,0}$ on U if and only if it is a function of the variables $z^{\kappa+2}, \dots, z^{2n}$, that is

$$\theta = \theta(z^{\kappa+2}, \dots, z^{2n}). \tag{4.11}$$

By using a convenient bump function ψ^i in each variable z^i $(\kappa + 2 \le i \le 2n)$, we obtain smooth functions $\theta_i := \psi^i \cdot z^i \in C^\infty(\mathcal{T}M)$ (no summation convention is used here), such that $\theta_i(v_0) = 1$, $\frac{d\theta_i}{dz^i}(v_0) = 1$ and $\operatorname{supp}(\theta_i) \subset U$. It is clear that

$$\theta_{\kappa+2},\ \dots\ , \theta_{2n} \tag{4.12}$$

are functionally independent $0-$homogeneous holonomy invariant functions on some neighbourhood $\widetilde{U} \subset U$ of v_0 and any elements of $\mathcal{H}_{S,0}$ can be expressed on $\widetilde{U}$ as their functional combination. The functions (4.12) can be used to "*modify*" the original Euler-Lagrange function E_o to obtain new elements of $\mathcal{E}_{S,2}$, functionally independent on $\widetilde{U}$.

Indeed, let $E_i := (1 + \theta_i)E_o$ for $\kappa + 2 \le i \le 2n$, and set $E_{\kappa+1} := E_o$. Since $1 + \theta_i$ are $0-$homogeneous and E_o is 2-homogeneous holonomy invariant functions we get that

$$E_{\kappa+1}, E_{\kappa+2}, \dots , E_{2n}, \tag{4.13}$$

are $2-$homogeneous holonomy invariant functions. Then, by Lemma 4.5, the elements of (4.13) are in $\mathcal{E}_{S,2}$. Moreover, by the construction we have $dE_i = d\big((1+\theta_i)E_o\big) = \frac{d\theta_i}{dz^i} E_o dz^i + (1+\theta_i) dE_o$ (with no summation on i). Hence,

$$(dE_i)_{v_0} = (dz^i)_{v_0} + \big(1 + \theta_i(v_0)\big)(dE_o)_{v_0}.$$

and taking (4.9) into account we get

$$\begin{aligned} dE_{\kappa+1}\wedge dE_{\kappa+2}\wedge\cdots\wedge dE_{2n}(v_0) &= \big(dE_o\wedge(dz^{\kappa+2}+\theta_{\kappa+2}dE_o)\wedge\cdots\wedge(dz^{2n}+\theta_{2n}dE_o)\big)_{v_0}\\ &= (dE_o\wedge dz^{\kappa+2}\wedge\cdots\wedge dz^{2n})_{v_0}\\ &= 2(\lambda E_o\, dz^{\kappa+1}\wedge dz^{\kappa+2}\wedge\cdots\wedge dz^{2n})_{v_0}\neq 0, \end{aligned}$$

that is, the functions (4.13) are functionally independent in some neighbourhood $\widehat{U}\subset\widetilde{U}$ of $v_0\in\mathcal{T}M$.

On the other hand, let us suppose that $E\in\mathcal{E}_{S,2}$ is a 2−homogeneous Euler-Lagrange function associated to S. Using Lemma 4.5, we get that $\theta=E/E_o$ is a 2−homogeneous holonomy invariant function. Then, θ has the form (4.11) on U and it can thus be expressed as a functional combination $\theta=\Psi(\theta_{\kappa+2},\dots,\theta_{2n})$. Since $E_o=E_{\kappa+1}$ we get

$$E=\Psi\left(\frac{E_{\kappa+2}}{E_{\kappa+1}},\dots,\frac{E_{2n}}{E_{\kappa+1}}\right)\cdot E_{\kappa+1}$$

showing that E is locally a functional combinations of the elements (4.13). □

Metrizability freedom.

Similar to the notion of variational freedom, one can introduce the *metrizability freedom* of a spray S showing how many functionally independent Finsler energy functions and hence how many essentially different Finsler metrics exist for S. To be more precise, let $\mathcal{E}^+_{S,2}$ be the set of Finler energy functions, that is the set of regular 2−homogeneous Lagrange function with (2.4) positive definite. Alike to (2.6) and (2.7) we set

Definition 4.8. Let S be a spray. If S is metrizable then its *metrizability freedom* is $m_s(\in\mathbb{N})$ where $m_s=rank\,(\mathcal{E}^+_{S,2})$. If S is non-variational then we set $m_s=0$.

We have

Proposition 4.9. *Let S be a metrizable spray such that the parallel translation with respect to the associated connection is regular. Then* $m_s=\operatorname{codim}\mathcal{D}_{\mathcal{H}}$.

Proof. Using the reasoning of Theorem 4.4 we can easily prove Proposition 4.9. We just remark that, using the notation introduced in the proof of Theorem 4.4, we have $E_0\in\mathcal{E}^+_{S,2}$ and for any $i=\kappa+2,\dots,2n$, a sufficiently small nonzero constant $c_i\in\mathbb{R}$ can be choosen for $E_i=(1+c_i\theta_i)E_o$ to remain positive definite. Hence with $E_o=E_{\kappa+1}$ we get $\{E_{\kappa+1},\,E_{\kappa+2},\,\dots,E_{2n}\}\subset\mathcal{E}^+_{S,2}$. Similar argument that we used in the proof of Theorem 4.4 shows that these elements are locally functionally independent and they locally generate $\mathcal{E}^+_{S,2}$ which proves the Proposition. □

5. Examples: Isotropic sprays

Let S be a spray and R its curvature tensor defined by (3.2). The *Jacobi endomorphism* Φ of the S is defined by

$$\Phi=i_SR. \tag{5.1}$$

The Ricci curvature, Ric, and the Ricci scalar, ρ are given by $\mathrm{Ric}=(n-1)\rho=R^i_i=\mathrm{Tr}(\Phi)$ [13].

Definition 5.1. A spray S is said to be *isotropic* if its Jacobi endomorphism has the form

$$\Phi=\rho J-\alpha\otimes C,$$

where $\rho\in C^\infty(\mathcal{T}M)$ is the Ricci scalar and α is a semi-basic 1-form on $\mathcal{T}M$.

Lemma 5.2. *For an isotropic sprays with non vanishing Ricci scalar one has* $\dim\mathcal{D}_{\mathcal{H}}\geq 2n-1$.

Proof. Let $X \in H\mathcal{T}M$ be a horizontal vector. We have

$$\Phi(X)=0 \iff \rho JX-\alpha(X)\mathcal{C}=0 \iff JX=\frac{i_X\alpha}{\rho}JS \iff X=\frac{i_X\alpha}{\rho}S. \tag{5.2}$$

By (5.2), $\ker\,\Phi \cap H\mathcal{T}M = Span\{S\}$ and, therefore, using the semi-basic property of Φ, we get $\ker\,\Phi = V\mathcal{T}M \oplus S$ and $\dim\ker\,\Phi = n+1$. Hence, we have $\dim(\mathrm{Im}\Phi) = 2n-(n+1) = n-1$. On the other hand, by (5.1), $\Phi(X)=(i_SR)(X)=R(S,X)$. Thus $\mathrm{Im}\,\Phi \subset \mathrm{Im}\,R$ and $\dim(\mathrm{Im}R) \geq n-1$. By Remark 3.2, the result follows. □

Proposition 5.3. *Let S be an isotropic spray on an n-dimensional manifold M with regular parallel translation. Then we have $\nu_{S,2} \in \{0,1,n\}$. More precisely we have the following possibilities:*

(a) $\nu_{S,2} = 0$ if and only if $R \neq 0$ and S is not variational (in this case $R \neq 0$);
(b) $\nu_{S,2} = 1$ if and only if $R \neq 0$ and S is variational;
(c) $\nu_{S,2} = n$, that is maximal, if and only if $R = 0$.

Proof. Let us first consider *(c)*. We remark that if $R = 0$, then the holonomy is trivial and S is Riemann and Finsler metrizable variational: an arbitrary Minowski norm extended through parallel translation defines a Finsler norm for S. Moreover, by using Remark 3.2 and Theorem 4.4, we have

$$\nu_{S,2} = n \;\Leftrightarrow\; \mathrm{codim}\,\mathcal{D}_{\mathcal{H}} = n \;\Leftrightarrow\; \dim\mathcal{D}_{\mathcal{H}} = n \;\Leftrightarrow\; R = 0.$$

(a) We have $\nu_{S,2} = 0$ if and only if S is not $h(2)-$variational. In that case we have necessarily $R \neq 0$. Using Theorem 2.3 we get that S cannot be variational.
(b) Let $\nu_{S,2} = 1$. Then S is $h(2)$-variational with an essentially unique $h(2)-$homogeneous regular Lagrange function. Then $\mathrm{codim}\,\mathcal{D}_{\mathcal{H}} < n$ and therefore $H\mathcal{T}M \subsetneq \mathcal{D}_{\mathcal{H}}$. Using Remark 3.2 we obtain that $R \neq 0$. Conversely, if S is variational and $R \neq 0$ then by Lemma 5.2, we have $\dim\mathcal{D}_{\mathcal{H}} \geq 2n-1$. On the other hand, Lemma 4.7 shows that $\mathcal{C} \notin \mathcal{D}_{\mathcal{H}}$ and therefore $\dim\mathcal{D}_{\mathcal{H}} \leq 2n-1$. From the two inequalities, we have $\dim\mathcal{D}_{\mathcal{H}} = 2n-1$ and hence $\mathrm{codim}\,\mathcal{D}_{\mathcal{H}} = 1$. □

Explicite examples.

Example 1 ($\nu_{S,2} = 0$, $\mathrm{codim}\,\mathcal{D}_{\mathcal{H}} = 0$).
Let $M = \{(x^1,x^2) \in \mathbb{R}^2 : x^2 > 0\}$ and S be the spray (2.2) given by the coefficents

$$G^1 := \varphi\, y^1 + \frac{y^1y^2}{2x^2}, \qquad G^2 := \varphi\, y^2 - \frac{(y^1)^2}{4},$$

where we used the notation $\varphi := \left(x^2(y^1)^2 + (y^2)^2\right)^{1/2}$. The spray S is isotropic and the coefficients of the nonlinear connection are given by

$$N^1_1 = \frac{y^2}{2x^2} + \varphi + \frac{x^2(y^1)^2}{\varphi}, \quad N^2_1 = -\frac{y^1}{2} + \frac{x^2y^1y^2}{\varphi}, \quad N^1_2 = \frac{y^1}{2x^2} + \frac{y^1y^2}{\varphi}, \quad N^2_2 = \varphi + \frac{(y^2)^2}{\varphi}.$$

The horizontal basis is $\{h_1,h_2\}$ where

$$h_1 = \frac{\partial}{\partial x^1} - \left(\frac{y^2}{2x^2} + \varphi + \frac{x^2(y^1)^2}{\varphi}\right)\frac{\partial}{\partial y^1} + \left(\frac{y^1}{2} - \frac{x^2y^1y^2}{\varphi}\right)\frac{\partial}{\partial y^2},$$
$$h_2 = \frac{\partial}{\partial x^2} - \left(\frac{y^1}{2x^2} + \frac{y^1y^2}{\varphi}\right)\frac{\partial}{\partial y^1} - \left(\varphi + \frac{(y^2)^2}{\varphi}\right)\frac{\partial}{\partial y^2}.$$

We have

$$v_1 := [h_1,h_2] = \frac{4(x^2)^2+1}{4(x^2)^2}\left(y^1\frac{\partial}{\partial y^2} - y^2\frac{\partial}{\partial y^1}\right)$$
$$v_2 := \big[[h_1,h_2],h_1\big] = \frac{4(x^2)^2+1}{4x^2\varphi}\left(y^1y^2\frac{\partial}{\partial y^1} + \left(\varphi^2 + (y^2)^2\right)\frac{\partial}{\partial y^2}\right).$$

Being v_1 and v_2 linearly independent we have $\mathcal{D}_{\mathcal{H}} = Span\{h_1,h_2,v_1,v_2\} = T\mathcal{T}M$. Consequently, $\mathcal{C} \in \mathcal{D}_{\mathcal{H}}$ and according to Lemma 4.7 the spray is not variational; that is $\nu_{S,2} = 0$.

Example 2 ($\mathcal{V}_{S,2}=0$, $\operatorname{codim}\mathcal{D}_{\mathcal{H}}>0$).
Let $M=\{(x^1,x^2)\in\mathbb{R}^2 : x^2>0\}$ and S the spray (2.2) given by the coefficients $G^1=\frac{(y^1)^2}{2x^2}$, $G^2=0$. The non zero coefficient of the non linear connection is $N^1_1=\frac{y^1}{x^2}$. The horizontal basis $\{h_1,h_2\}$ and their commutator are

$$h_1=\frac{\partial}{\partial x^1}-\frac{y^1}{x^2}\frac{\partial}{\partial y^1},\qquad h_2=\frac{\partial}{\partial x^2},\qquad v:=[h_1,h_2]=-\frac{y^1}{(x^2)^2}\frac{\partial}{\partial y^1}.$$

One has $\mathcal{D}_{\mathcal{H}}=Span\{h_1,h_2,v\}$, $\dim\mathcal{D}_{\mathcal{H}}=3$ and $\operatorname{codim}\mathcal{D}_{\mathcal{H}}=1$. For any holonomy invariant 2-homogeneous function $E\in\mathcal{H}_{S,2}$ we have $\mathcal{L}_{h_1}E=\mathcal{L}_{h_2}E=\mathcal{L}_vE=0$. From the last equation we get $\frac{\partial E}{\partial y^1}=0$ and threfore E cannot be a regular Lagrange function. From Lemma 4.3, it follows that S has no regular Euler-Lagrange function and, therefore, it cannot be variational.

Example 3 ($\mathcal{V}_{S,2}=1$).
Let us consider on the standard unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ and the spray (2.2) where $G^i=-\dfrac{\mu\langle x,y\rangle}{1+\mu\,|x|^2}y^i$ with $\mu\in\mathbb{R}\setminus\{0\}$. The curvature of the spray is non zero and isotropic. The spray is metrisable and hence variational: it is the geodesic spray of the Riemannian energy function

$$E_\mu=\frac{1}{2}\frac{\mu(|x|^2|y|^2-\langle x,y\rangle^2)+|y|^2}{(1+\mu|x|^2)^2}\tag{5.3}$$

of constant flag curvature $\mu\neq 0$. From Proposition 5.3, we have $\mathcal{V}_{S,2}=1$. Hence (5.3) is the essentially unique energy function corresponding to the given spray.

Example 4 ($\mathcal{V}_{S,2}$ is maximal).
One can consider the trivial example where $M=\mathbb{R}^n$ and the spray (2.2) where $G^i=0$. In this case the parallel translation is regular and the holonomy group is trivial. Hence we have $\mathcal{V}_{S,2}=n$. We prefer to give also another, not so obvious example: Let $\mathbb{B}^n\subset\mathbb{R}^n$ be the standard unit ball and S the spray with

$$G^i=-\frac{\langle a,y\rangle}{1+\langle a,x\rangle}y^i,\tag{5.4}$$

where $a\in\mathbb{R}^n$ is a constant vector with $|a|<1$. Since $R=0$, then $\mathcal{D}_{\mathcal{H}}=H\mathcal{T}M$, the horizontal distribution. Hence, by Theorem 4.4, the metric freedom is maximal. We remark that S.S. Chern and Z. Shen investigated in [4] the family of Riemannian metrics associated to the norms

$$F_a=\frac{\sqrt{1-|a|^2}}{(1+\langle a,x\rangle)^2}\sqrt{|y|^2-\frac{2\langle a,y\rangle\langle x,y\rangle}{1+\langle a,x\rangle}-\frac{(1-|x|^2)\langle a,y\rangle^2}{1+\langle a,x\rangle}}.\tag{5.5}$$

The geodesic equation of (5.5) is (5.4), but one can find other generating Finsler metrics too. Indeed, putting $z^i=((1+\langle a,x\rangle)y^i-\langle a,y\rangle x^i)/(1+\langle a,x\rangle)^2$ and considering a 1-homogeneous function $\phi\colon\mathbb{R}^n\to\mathbb{R}$ we get

$$F_\phi(x,y)=\phi\big(z^1(x,y),\dots,z^n(x,y)\big)\tag{5.6}$$

such that $E_\phi=\frac{1}{2}F_\phi^2$ is a (not necessarily regular) element of $\mathcal{E}_{S,2}$. Therefore, if F_ϕ satisfies the regularity condition (2.4), then it is a projectively flat Finsler mertic of zero flag curvature with geodesic spray given by (5.4). The family (5.5) can be considered as a special case of (5.6) by choosing $\phi(z)=\big(\langle z,z\rangle-\langle a,z\rangle^2\big)^{1/2}$.

6. Open problems

In this paper we considered sprays and investigated how many essentially different $2-$homogeneous regular Lagrange functions and $h(2)$-variational principles exist for a given spray. We obtained the formula (4.4) in the metrizable case. The first problem would be

Problem 1. Determine the $h(2)-$variational freedom without the metrizability assumption.

For different degrees of homogeneity, we can also consider the following:

Problem 2. Determine how many essentially different k–homogeneous regular Lagrange functions and variational principles exist for a given spray.

The most interesting challenge might be the general case:

Problem 3. Determine how many essentially different (not necessary homogeneous) variational principles exist for a given spray.

This last problem can be hard to solve, since in the non-homogeneous case there is no simple correspondence between Euler-Lagrange functions and holonomy invariant functions.

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Salah G. Elgendi, Institute of Mathematics, University of Debrecen, Debrecen, Hungary
E-mail address: `salah.ali@fsci.bu.edu.eg`

Zoltán Muzsnay, Institute of Mathematics, University of Debrecen, Debrecen, Hungary
E-mail address: `muzsnay@science.unideb.hu`
URL: `http://math.unideb.hu/muzsnay-zoltan`